Upstream differencing for multiphase flow in reservoir simulation

Yann Brenier, Jérôme Jaffré

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 1145

Programme 7
Calcul Scientifique,
Logiciels Numériques et Ingénierie Assistée

UPSTREAM DIFFERENCING FOR MULTIPHASE FLOW IN RESERVOIR SIMULATION

Yann BRENIER
Jerôme JAFFRE

Décembre 1989

RR.1145

UPSTREAM DIFFERENCING FOR MULTIPHASE FLOW IN RESERVOIR SIMULATION

DECENTRAGE AMONT POUR LES ECOULEMENTS MULTIPHASIQUES EN SIMULATION DE RESERVOIRS.

Yann Brenier, Jérôme Jaffré

INRIA, B.P. 105, 78153 Le Chesnay cédex, France

Abstract

Upstream weighting for multiphase flow in reservoir simulation is analyzed. The associated numerical flux is shown to be well defined, monotone, Lipschitz-continuous and consistent . In the case of a two-phase flow the corresponding numerical scheme is convergent and the numerical flux is compared to that of Godunov and Engquist and Osher. Finally simple way to obtain a higher order scheme is outlined.

Résumé

Le décentrage amont pour des écoulements multiphasiques en simulation de réservoir est analysé. On montre que le flux numérique est bien défini, monotone, Lipschitzien et consistant. Dans le cas d'un écoulement de fluides à deux phases, le schéma numérique correspondant est convergent et le flux numérique est comparé à celui de Godunov et de Engquist et Osher. Finalement on décrit brièvement comment obtenir simplement un schéma d'ordre supérieur.

Key words: Conservation laws, multiphase flow, reservoir simulation, Riemann solver, upstream weighting.

Mots clés: Lois de conservation , écoulement multiphasique, simulation de réservoir, solveur de Riemann, décentrage amont.

I. Introduction

Upstream differencing is widely used in computational fluid dynamics to design numerical schemes for hyperbolic conservation laws [6]. Upstream differencing is also very popular for the simulation of multiphase flow in petroleum reservoirs [8] but there it denotes a numerical scheme different from those used in classical CFD and obtained from simple physical considerations. In the case of incompressible two-phase flow this scheme has been already studied and cast into the general frame of monotone finite difference schemes [10]. In this paper we show that the calculation of the multiphase upstream weighted numerical fluxes is well defined even in the case of more than two phases and we study the properties of these fluxes. In the case of two-phase flow, convergence results are stated as in [10] and the two-phase upstream weighted numerical flux is compared to standard ones used in classical CFD. Finally we present a simple way to design higher order methods which would require a minimal amount of change in existing programs actually used in reservoir simulation.

We consider n immiscible fluids flowing in a one-dimensional medium. We neglect capillary effects, so a unique pressure p is defined for the multiphase flow. Also we assume that the phases are incompressible. For the fluid ℓ, $1 \le \ell \le n$, we denote by S_ℓ the saturation, ρ_ℓ the density, k_ℓ the mobility and φ_ℓ the flow rate. Then the flow is governed by the following equations derived from mass conservation and Darcy's law

$$(1.1) \qquad \phi \frac{\partial S_\ell}{\partial t} + \frac{\partial}{\partial x} \varphi_\ell = 0, \qquad \ell = 1,\dots,n,$$

$$(1.2) \qquad \varphi_\ell = -Kk_\ell\left(\frac{\partial p}{\partial x} - g_\ell\right), \qquad \ell = 1,\dots,n.$$

ϕ and K are respectively the porosity and the absolute permeability multiplied by the cross-sectorial area, and g_ℓ is the gravity term of Darcy's law

$$g_\ell = \rho_\ell\, g \frac{\partial z}{\partial x}$$

where g is the gravitation constant and z is the depth at the position x.

Introduce the total flow rate $q = \sum_\ell \varphi_\ell$. As $\sum_\ell S_\ell = 1$, by summing equation (1.1) and by using the incompressibility condition we obtain $\partial/\partial x\, q = 0$, which implies that the total flow rate q is constant with respect to the space variable.

To express φ_ℓ in terms of q instead of p, we sum equations (1.2) in order to eliminate p :

$$- K \frac{\partial p}{\partial x} = \left(\sum_j k_j\right)^{-1}\left(q+K\sum_j g_j k_j\right).$$

Plugging this expression in (1.2) we obtain

$$\varphi_\ell = \left(\sum_j k_j\right)^{-1} k_\ell \left(q+K\sum_j (g_\ell - g_j)k_j\right).$$

Therefore, as the mobilities k_j are functions of $S = (S_1,\dots,S_n)$, equations (1.1), (1.2) can be rewritten as the following system of equations for the saturations

$$(1.3) \qquad \phi \frac{\partial S_\ell}{\partial t} + \frac{\partial}{\partial x} \varphi_\ell(S) = 0,$$

$$(1.4) \qquad \varphi_\ell(S) = \left(\sum_j k_j(S)\right)^{-1} k_\ell(S) \left(q + K \sum_j (g_\ell - g_j) k_j(S)\right)$$

where q is given by some boundary condition.

In reservoir simulation such a system of equations is usually discretized as follows [8]. Denote $x_{i+1/2}$, $i \in \mathbb{Z}$, the discretization points in space and $S_{\ell,i}(t)$ the constant value of the approximate saturation on the interval $(x_{i-1/2}, x_{i+1/2})$ with $h_i = x_{i+1/2} - x_{i+1/2}$. Leaving out time discretization, equations (1.3), (1.4) are semi-discretized in space by

$$(1.5) \qquad \phi_i \frac{dS_{\ell,i}}{dt} + \frac{1}{h}\left(\varphi_{\ell,i+1/2} - \varphi_{\ell,i-1/2}\right) = 0$$

where the numerical flux $\varphi_{\ell,i+1/2}$ is the approximiate flow-rate of the phase ℓ across $x_{i+1/2}$ and is given by

$$(1.6) \qquad \varphi_{\ell,i+1/2} = \left(\sum_j k_{j,i+1/2}\right)^{-1} k_{\ell,i+1/2} \left(q + K_{i+1/2} \sum_j (g_\ell - g_j)\, k_{j,i+1/2}\right).$$

The mobilities $k_{\ell,i+1/2}$, $1 \le \ell \le n$, are calculated using the upstream saturations with respect to the flow of the phase ℓ :

$$(1.7) \qquad k_{\ell,i+1/2} = \begin{cases} k_\ell(S_i) & \text{if } q + K_{i+1/2} \sum_j (g_\ell - g_j)\, k_{j,i+1/2} > 0, \\ k_\ell(S_{i+1}) & \text{otherwise.} \end{cases}$$

We note that expressions (1.6), (1.7) do not yield an explicit calculation of the numerical fluxes $\varphi_{\ell,i+1/2}$ since they depend on the mobilities which themselves depend on the sign of the numerical fluxes.

Therefore our first task is to show that one can derive explicit formulas for the multiphase upstream weighted numerical fluxes ; this is done by ordering the phases with increasing weights (section II). Then we give some regularity properties of these numerical fluxes (section III). In the case of two-phase flow these properties imply convergence of the associated numerical scheme and the two-phase upstream weighted numerical flux is compared with the Godunov and Engquist-Osher ones (section IV). Finally we present a higher order method which preserves the calculation of numerical fluxes, thus minimizing the amount of work necessary to modify programs actually used in reservoir simulation (section V).

II. Explicit calculation of upstream weighted numerical fluxes

To simplify the notations we drop in (1.6), (1.7) the index i+1/2 and denote $a = S_i$, $b = S_{i+1/2}$. Then (1.6), (1.7) are rewritten as, for $1 \le \ell \le n$,

$$(2.1) \qquad \overset{*}{\varphi}_\ell = \left(\sum_j \overset{*}{k}_\ell\right)^{-1} \overset{*}{k}_\ell \left(q + K \sum_j (g_\ell - g_j)\, \overset{*}{k}_\ell\right),$$

$$(2.2) \qquad \overset{*}{k}_\ell = \begin{cases} k_\ell(a) & \text{if } q + K \sum_j (g_\ell - g_j)\, \overset{*}{k}_\ell > 0 \\ k_\ell(b) & \text{otherwise} \end{cases}$$

We are going to give an equivalent expression for (2.2) which does not assume that we know the sign of the numerical fluxes.

Let us order the phases with increasing weights :

(2.3) $$g_1 \le g_2 \le \dots \le g_n$$

and consider the quantities :

(2.4) $$\theta_\ell = q+K\left\{\sum_{j<\ell}(g_\ell-g_j)k_j(b) + \sum_{j>\ell}(g_\ell-g_j)k_j(a)\right\}$$

Lemma 2.1

Once the phases have been indexed with increasing weights, the quantities θ_ℓ defined in (2.4) are increasing with the index ℓ of the phases, $1 \le \ell \le n$.

Proof

Since $g_\ell = g_j$ for $j=\ell$, we have

$$\theta_\ell = q+K\left\{\sum_{j<\ell}(g_\ell-g_j)k_j(b) + \sum_{j\ge\ell}(g_\ell-g_j)k_j(a)\right\}$$

and since $g_{\ell-1} = g_j$ for $j=\ell-1$, we can also write

$$\theta_{\ell-1} = q+K\left\{\sum_{j<\ell}(g_{\ell-1}-g_j)k_j(b) + \sum_{j\ge\ell}(g_{\ell-1}-g_j)k_j(a)\right\}.$$

Therefore

$$\theta_\ell-\theta_{\ell-1} = K\,(g_\ell-g_{\ell-1})\left(\sum_{j<\ell}k_j(b) + \sum_{j\ge\ell}k_j(a)\right).$$

Since the mobilities k_j and the absolute permeabilities K are positive quantities, by using (2.3) we obtain

$$\theta_\ell - \theta_{\ell-1} \ge 0.$$

Introduce now the integer $r \in \{0,\dots,n+1\}$ such that :

(2.5) $$r = \begin{cases} 0, \text{ if } \theta_\ell > 0 \text{ for } 1 \le \ell \le n, \\ \text{largest } \ell \text{ such that } \theta_\ell \le 0, \text{ otherwise.} \end{cases}$$

This definition makes sense since the θ_ℓ 's form an increasing sequence.

The main result of this section is the following theorem.

Theorem 2.1

Expression (2.2) for the upstream mobilities is equivalent to

(2.6) $$k_\ell^* = \begin{cases} k_\ell(a) \text{ if } \ell > r, \\ k_\ell(b) \text{ if } \ell \le r, \end{cases}$$

for $1 \le \ell \le n$, *where r is defined by (2.5).*

Theorem 2.1 yields a simple algorithm to calculate the numerical fluxes. Once the phases have been indexed with increasing weights,

i) calculate the quantities θ_ℓ given by (2.4),

ii) determine r satisfying (2.5),

iii) calculate the mobilities from (2.6),

iv) plug the mobilities into (2.1) to obtain the numerical fluxes.

Proof of theorem 2.1

Whatever the way the mobilities k^*_j are calculated we denote :

$$\delta_\ell = q + K \sum_j (g_\ell - g_j)\, k_j^*, \quad 1 \le \ell \le n. \tag{2.7}$$

Since

$$\delta_\ell - \delta_{\ell-1} = K \sum_j (g_\ell - g_{\ell-1})\, k_j^*,$$

we deduce from (2.3) and the positivity of the mobilities and of the absolute permeability that the δ_ℓ 's form a sequence increasing with ℓ.

First let us assume that the mobilities are calculated with (2.6). Then :

$$\delta_\ell = q + K \left\{ \sum_{j \le r} (g_\ell - g_j)\, k_j(b) + \sum_{j > r} (g_\ell - g_j)\, k_j(a) \right\}.$$

On the other hand, since $g_\ell = g_j$ for $j=\ell$, θ_ℓ defined in (2.4) can be written in two ways :

$$\begin{aligned}\theta_\ell &= q + K \left\{ \sum_{j \le \ell} (g_\ell - g_j)\, k_j(b) + \sum_{j > \ell} (g_\ell - g_j)\, k_j(a) \right\} \\ &= q + K \left\{ \sum_{j \le \ell-1} (g_\ell - g_j)\, k_j(b) + \sum_{j > \ell-1} (g_\ell - g_j)\, k_j(a) \right\}.\end{aligned}$$

Comparing expressions for δ_ℓ and θ_ℓ , we obtain

$$\theta_r = \delta_r, \qquad \theta_{r+1} = \delta_{r+1}.$$

Since δ_ℓ , as well as θ_ℓ , is an increasing sequence, we obtain the following equivalences

$$\ell \le r \Leftrightarrow \theta_\ell \le 0 \Leftrightarrow \delta_\ell \le 0,$$

which shows that (2.6) implies (2.2).

Conversely assume that the mobilities are calculated with (2.2). Since the δ_ℓ 's form an increasing sequence, there exists an integer $m \in \{0,\dots,n+1\}$ such that :

$$\delta_\ell \le 0 \Leftrightarrow \ell \le m.$$

Then the mobilities given by (2.2) are :

$$k_\ell^* = \begin{cases} k_\ell(a) & \text{if} \quad \ell > m, \\ k_\ell(b) & \text{if} \quad \ell \le m, \end{cases}$$

and δ_ℓ can be written as :

$$\delta_\ell = q + K \left\{ \sum_{j \le m} (g_\ell - g_j) k_j(b) + \sum_{j > m} (g_\ell - g_j) k_j(a) \right\}.$$

Comparing with expression (2.8) for θ_ℓ we find :

$$\theta_m = \delta_m, \qquad \theta_{m+1} = \delta_{m+1}.$$

Since θ_ℓ is also an increasing sequence, and using (2.5), we can write :

$$\delta_\ell \le 0 \Leftrightarrow \ell \le m \Leftrightarrow \theta_\ell \le 0 \Leftrightarrow \ell \le r$$

which terminates the proof of theorem 2.1.

III. Properties of upstream weighted numerical fluxes

In this section we shall show some regularity, monotonicity and consistency properties for the numerical fluxes.

Theorem 3.1

Assume that the mobilities are continuous functions of the saturations and that their partial derivatives with respect to each saturation are bounded. Then the numerical fluxes defined by (2.1), (2.2) or equivalently by (2.1), (2.6) are Lipschitz-continuous functions.

Proof

It is clear that the numerical fluxes are piecewise regular functions and where they are regular they have the same regularity as the mobilities k_ℓ. However, a priori, they could be discontinuous along the lines $\theta_\ell(a,b)=0$. These lines can touch each other but cannot cross each other since the sequence $\theta_\ell(a,b)$ is increasing with ℓ (see lemma 2.1).

Let us consider such a line, say $\ell=m$,

$$R_m^0=\left\{(a,b)\in [0,1]^{2n} \mid \theta_m(a,b)=0\right\}.$$

Such a line divides the cube $[0,1]^{2n}$ into two parts :

$$R_m^+=\left\{(a,b)\in [0,1]^n\times[0,1]^n \mid \theta_m(a,b)>0\right\},$$

$$R_m^-=\left\{(a,b)\in [0,1]^n\times[0,1]^n \mid \theta_m(a,b)<0\right\}.$$

Consider a point $(\bar{a},\bar{b})\in R^0$ where we are going to show that the numerical fluxes have the regularity given in theorem 3.1. For simplicity, we assume that $\theta_\ell(\bar{a},\bar{b})\neq 0$ for $\ell\neq m$, that is we assume that no other line $R^0{}_\ell$ is touching $R^0{}_m$. Then there is a small neighborhood $\mathcal{V}$ of $(\bar{a},\bar{b})$ such that :

$$\theta_\ell(a,b)\le 0 \Leftrightarrow \ell\le m-1 \quad \text{for } (a,b)\in\mathcal{V}\cap R_m^+,$$

$$\theta_\ell(a,b)\le 0 \Leftrightarrow \ell\le m \quad \text{for } (a,b)\in\mathcal{V}\cap R_m^-,$$

or from (2.6),

$$\text{(3.1)}\qquad k_\ell^* = \begin{cases} k_\ell(a) \text{ if } \ell>m \\ k_\ell(b) \text{ if } \ell<m \end{cases}, \text{ for } (a,b)\in\mathcal{V}$$

and

$$\text{(3.2)}\qquad k_m^* = \begin{cases} k_m(a) \text{ for } (a,b)\in\mathcal{V}\cap R_m^+, \\ k_m(b) \text{ for } (a,b)\in\mathcal{V}\cap R_m^-. \end{cases}$$

Thus across $R^0{}_m$ one has to switch from the second argument b to the first argument a to calculate $k^*{}_m$. Therefore across $R^0{}_m$, all the numerical mobilities $k^*{}_\ell$, $\ell\neq m$, are as regular as the mobilities k_ℓ, except $k^*{}_m$ which is discontinuous.

From (2.1), (2.7), the numerical fluxes can be written as

$$\varphi_\ell^* = \left(\sum_j k_j^*\right)^{-1} k_\ell^*\delta_\ell.$$

Since

(3.3) $$\delta_m = q + K \sum_j (g_m - g_j)k_j^* = \delta_\ell + K (g_m - g_\ell) \sum_j k_j^*$$

We obtain

$$\varphi_\ell^* = \left(\sum_j k_j^*\right)^{-1} k_\ell^* \left[K \sum_j (g_\ell - g_m) k_j^* + \delta_m\right]$$

$$= K(g_\ell - g_m) k_\ell^* + \left(\sum_j k_j^*\right)^{-1} k_\ell^* \delta_m.$$

When (a,b) tends to $(\bar{a}, \bar{b}) \in R^0_m$, then $\delta_m = \theta_m \to 0$. Therefore $\varphi^*_m \to 0$ and for $\ell \neq m$ $\varphi^*_\ell \to (g_\ell - g_m)k^*_\ell$. Consequently φ^*_ℓ is continuous for all $\ell = 1,\dots,n$.

Now we turn to the derivatives of φ^*_ℓ. By differentiating (2.1), we obtain :

(3.4) $$\partial\varphi_\ell^* = \frac{\left(\sum_j k_j^*\right)\partial k_\ell^* - \left(\sum_j \partial k_j^*\right) k_\ell^*}{\left(\sum_j k_j^*\right)^2} \delta_\ell + \frac{k_\ell^*}{\left(\sum_j k_j^*\right)} \left(K \sum_j (g_\ell - g_j)\partial k_j^*\right).$$

Here ∂ denotes any partial derivative with respect to a component of a or of b.

Since $\sum_j k_j^*$ is always strictly positive, it is clear from (3.4) that the derivatives are all bounded, so the numerical fluxes are Lipschitz-continuous.

Remark 3.1

Numerical fluxes are not more regular than Lipschitz-continuous. Indeed let us check that their derivatives are discontinuous. From (3.3), (3.4) and rearranging terms, it follows :

$$\partial\varphi_\ell^* = \partial k_\ell^* \left[K(g_\ell - g_m) + \frac{\delta_m}{\sum_j k_j^*}\right] - \frac{\left(\sum_j \partial k_j^*\right)}{\left(\sum_j k_j^*\right)} k_\ell^* \left[K (g_\ell - g_m) + \frac{\delta_m}{\sum_j k_j^*}\right] + \frac{k_\ell^*}{\sum_j k_j^*} \left(K \sum_j (g_\ell - g_j)\partial k_j^*\right)$$

$$= \partial k_\ell^* K(g_\ell - g_m) + \left[\partial k_\ell^* - \frac{\sum_j \partial k_j^*}{\left(\sum_j k_j^*\right)} k_\ell^*\right] \frac{\delta_m}{\left(\sum_j k_j^*\right)} + \frac{k_\ell^*}{\sum_j k_j^*} \left[K \sum_j (g_m - g_j)\partial k_j^*\right].$$

The first two terms of the sum are continuous across R^0_m but the third one is discontinuous : for example, for $\ell < m$, when considering the partial derivative with respect to the i^{th} component of b denoted b_i, inside R^+_m this term is equal to

$$\frac{k_\ell(b)}{\sum_{j=1}^{m} k_j(b) + \sum_{j=m+1}^{n} k_j(a)} \left[K \sum_{j=1}^{m-1} (g_m - g_j)\partial_{b_i} k_j(b)\right]$$

while inside R^-_m it is equal to

$$\frac{k_\ell(b)}{\sum_{j=1}^{m-1} k_j(b) + \sum_{j=m}^{n} k_j(a)} \left[K \sum_{j=1}^{m-1} (g_m - g_j)\partial_{b_\ell} k_j(b) \right]$$

and these two quantities are different as long as a is different from b. Therefore the derivatives of the numerical fluxes are discontinuous across the line $\theta_m(a,b)=0$.

Now we give monotonicity properties for the upstream weighted numerical fluxes.

Theorem 3.2

Assume that the mobility of the phase ℓ *is increasing with the saturation of the same phase and decreasing with the saturations of the other phases, for* $\ell=1,\dots,n$. *Then the numerical flux* $\varphi^*_\ell(a,b)$ *defined by (2.1),(2.2) or (2.1),(2.6) is an increasing (resp.decreasing) function of the* ℓ^{th} *component of the first (resp.second) argument* a *(resp.*b*).*

Proof

We shall show this theorem by studying the adequate partial derivatives.

Rearranging terms in (3.4) we can write the derivatives in a different form :

$$\partial\varphi^*_\ell = \left(\sum_j k^*_j\right)^{-2} \left\{ \left[\left(\sum_j k^*_j\right)\partial k^*_\ell - \left(\sum_j \partial k^*_j\right) k^*_\ell \right]\delta_\ell + k^*_\ell \left(K \sum_j (g_\ell - g_j)\partial k^*_j \right)\left(\sum_j k^*_j\right) \right\}$$

$$= \left(\sum_j k^*_j\right)^{-2}\left[\left(\sum_{j\neq\ell} k^*_j\right)\partial k^*_\ell\, \delta_\ell - k^*_\ell \sum_{j\neq\ell}\partial k^*_j \left(\delta_\ell - K\,(g_\ell - g_j)\left(\sum_{j'} k^*_{j'}\right)\right)\right].$$

Finally, we obtain

$$\text{(3.5)} \qquad \partial\varphi^*_\ell = \left(\sum_j k^*_j\right)^{-2}\left[\left(\sum_{j\neq\ell} k^*_j\right)\partial k^*_\ell\, \delta_\ell - k^*_\ell \left(\sum_{j\neq\ell}\partial k^*_j \delta_j\right)\right].$$

Assume for instance that $\ell \le r$. Then we have

$$\text{for } j \le r,\ \delta_j \le 0,\ k^*_j = k_j(b),\ \partial_{a_\ell} k^*_j = 0,\ \partial_{b_\ell} k^*_j = \partial_\ell k_j,$$

$$\text{for } j > r,\ \delta_j > 0,\ k^*_j = k_j(a),\ \partial_{a_\ell} k^*_j = \partial_\ell k_j,\ \partial_{b_\ell} k^*_j = 0.$$

Therefore

$$\partial_{b_\ell}\varphi^*_\ell = \left(\sum_j k^*_j\right)^{-2}\left[\left(\sum_{j\neq\ell} k^*_j\right)\partial_\ell k_\ell(b)\, \delta_\ell - k_\ell(b) \sum_{\substack{j=1 \\ j\neq\ell}}^{r} (\partial_\ell k_j(b)\, \delta_j)\right] \le 0$$

since $\partial_\ell k_\ell(b) \ge 0$ and $\partial_\ell k_j(b) \le 0$ for $\ell \ne j$.

Similarly

$$\partial_{a_\ell}\varphi^*_\ell = \left(\sum_j k^*_j\right)^{-2}\left[-k_\ell(b) \sum_{j=r+1}^{n} (\partial_\ell k_j(a)\, \delta_j \right] \ge 0$$

The case $\ell > r$ is solved analoguously.

We terminate this section by giving consistency properties for the upstream weighted numerical fluxes.

Theorem 3.3

The numerical flux φ^*_ℓ defined by (2.1),(2.2) or (2.1),(2.6) is consistent, i.e.

$$\overset{*}{\varphi}_\ell(a,a) = \varphi_\ell(a), \quad \ell=1,\dots,n.$$

This property is obvious from the definition of the numerical fluxes.

IV. The case of two-phase flow

When considering only two phases (n=2), the system of conservation laws reduces to one scalar conservation law to which we can apply the general theory of approximation of scalar non linear conservations laws.

The two-phase model can be written as :

$$\frac{\partial S}{\partial t} + \frac{\partial f(S)}{\partial x} = 0, \tag{4.1}$$

$$f = \varphi_2 = \frac{k_2}{k_1+k_2}(q+K(g_2-g_1)k_1), \tag{4.2}$$

where $S=S_2$, so the mobility k_1 (resp. k_2) is a decreasing (resp increasing) function of S. We assume that $q\geq 0$ and that the phases have been numbered with increasing weight ($g_2 \geq g_1$).

We approximate the two-phase model by the first order conservative scheme

$$\phi_i\,(S_i^{n+1}-S_i^n) + \frac{\Delta t}{h}(\varphi^n_{i+1/2}-\varphi^n_{i-1/2}) = 0, \tag{4.3}$$

$$\varphi^n_{1+1/2} = (k^n_{1,i+1/2}+k^n_{2,i+1/2})^{-1}\,k^n_{2,i+1/2}\,(q+K_{i+1/2}(g_2-g_1)k^n_{2,i+1/2}) \tag{4.4}$$

where the mobilities at $x_{i+1/2}$ are defined implicitely as

$$\begin{cases} k^n_{1,i+1/2} = \begin{cases} k_1(S_i^n) & \text{if } q+K_{i+1/2}(g_1-g_2)k^n_{2,i+1/2} > 0, \\ k_1(S^n_{i+1}) & \text{otherwise,} \end{cases} \\ k^n_{2,i+1/2} = \begin{cases} k_2(S_i^n) & \text{if } q+K_{i+1/2}(g_2-g_1)k^n_{1,i+1/2} > 0, \\ k_2(S^n_{i+1}) & \text{otherwise,} \end{cases} \end{cases} \tag{4.5}$$

As shown in section II, this implicit definition is equivalent to the following explicit definition. Given

$$\theta_1 = q+K_{i+1/2}(g_1-g_2)k_2(S^n_{i+1}), \quad \theta_2 = q+K_{i+1/2}(g_2-g_i)k_1(S_i^n), \tag{4.6}$$

then

$$\begin{cases} k^n_{1,i+1/2} = k_1(S_i^n) \text{ and } k^n_{2,i+1/2} = k_2(S_i^n) & \text{if } 0\leq\theta_1\leq\theta_2, \\ k^n_{1,i+1/2} = k_1(S^n_{i+1}) \text{ and } k^n_{2,i+1/2} = k_2(S_i^n) & \text{if } \theta_1\leq 0\leq\theta_2. \end{cases} \tag{4.7}$$

From theorems 3.1, 3.2, 3.3, we can apply a general theorem on convergence of monotone schemes [7],[3].

Theorem 4.1

Assuming that the mobility k_2 *is increasing and the mobility* k_1 *is decreasing, and that they are both differentiable functions with bounded derivatives, then the numerical scheme (4.3), (4.4) and (4.5) (or equivalently (4.3), (4.4), (4.6), (4.7)) is monotone and its solution converges to the entropy solution of the two-phase equations (4.1), (4.2) when* Δt *and* h *tend to* 0 *while satisfying the CFL condition :*

$$\operatorname*{Max}_{0\leq S_{i-1},S_i,S_{i+1}\leq 1}\left[\frac{\partial \overset{n}{\varphi}_{i+1/2}}{\partial a}(S_i,S_{i+1}) - \frac{\partial \overset{n}{\varphi}_{i-1/2}}{\partial b}(S_{i-1},S_i)\right]\frac{1}{\phi_i}\,\frac{\Delta t}{h} \leq 1 \;\blacksquare$$

This CFL condition has been written out by Sammon [10]. A nonuniform spatial mesh version of this theorem could be derived from [11].

Equations (4.4), (4.5), or equivalently (4.4), (4.6), (4.7), define a numerical flux for functions of the particular form (4.2) with k_1 decreasing and k_2 increasing. Therefore it is of some interest to compare this numerical flux to others widely used in other fields than reservoir simulations, as Godunov's [5] and Engquist and Osher's [4] numerical fluxes.

Sticking to notations of section III, the latter numerical flux functions are respectively

$$\text{(4.8)} \qquad F^{G}(a,b) = \begin{cases} \operatorname*{Min}_{s\in[a,b]} f(s) & \text{if} \quad a\leq b, \\ \operatorname*{Max}_{s\in[a,b]} f(s) & \text{if} \quad a\geq b, \end{cases}$$

$$\text{(4.9)} \qquad F^{EO}(a,b) = \frac{1}{2}\left[f(a) + f(b) - \int_a^b |f'(s)|\,ds\right].$$

For the numerical flux function associated to the upstream weighted numerical scheme, we introduce the quantities

$$\text{(4.10)} \qquad \theta_1 = q + K(g_1-g_2)k_2(b), \qquad \theta_2 = q + K(g_2-g_1)k_1(b),$$

and it is defined as

$$\text{(4.11)} \qquad F^{UW}(a,b) = \frac{\overset{*}{k}_2}{\overset{*}{k}_1+\overset{*}{k}_2}(q+K(g_2-g_1)\overset{*}{k}_1),$$

where :

$$\text{(4.12)} \qquad \begin{cases} \overset{*}{k}_1 = k_1(a),\ \overset{*}{k}_2 = k_2(a) & \text{if} \quad 0\leq\theta_1\leq\theta_2, \\ \overset{*}{k}_1 = k_1(b),\ \overset{*}{k}_2 = k_2(a) & \text{if} \quad \theta_1\leq 0\leq\theta_2. \end{cases}$$

These flux functions can be compared with respect to their regularity. Theorem 3.1 and remark 3.1 tell us that the upstream weighted and Godunov flux functions have the same regularity (Lipschitz-continuity) while the Engquist-Osher function is more regular (C^1-continuity).

The three flux functions can be also compared with respect to the amount of viscosity present in the associated monotone conservative scheme, and this amount can be measured in terms of the viscosity coefficient [12] :

$$Q(a,b) = \frac{\Delta t}{h} \; \frac{f(a) + f(b) - F(a,b)}{b - a}.$$

The Godunov viscosity coefficient is the smallest possible to ensure convergence of the associated monotone numerical scheme. Therefore the two-phase upstream weighted and Engquist-Osher viscosities coefficients are larger and we are left with the problem to compare them. From the expression of the viscosity coefficient, it is clear that this comparison boils down to compare the numerical flux function themselves : the scheme with the smaller flux function when $b<a$ and with the larger one when $b>a$ is the less viscous one.

Let us consider the case $b>a$, the other one being similar to study. We shall use the derivative of f with respect to S

$$f' = \frac{1}{(k_1 - k_2)^2} \left\{ k_1 k_2' \,(q + K(g_2 - g_1)k_1) - k_2 k_1' (q + K(g_1 - g_2)k_2) \right\}.$$

Introduce also the quantities

$$\eta_1 = q + K(g_1 - g_2)k_2(a), \qquad \eta_2 = q + K(g_2 - g_1)k_1(b)$$

Since $b>a$, $q \geq 0$, $g_2 \geq g_1$, by comparing with (4.10), we obtain

$$\theta_1 \leq \eta_1 \leq \eta_2 \leq \theta_2, \qquad 0 \leq \eta_2 \leq \theta_2 .$$

According to (4.12), we must consider two cases.

Let us assume first that $0 \leq \theta_1 \leq \eta_1 \leq \eta_2 \leq \theta_2$. Then, on one hand, from (4.12), we obtain

$$F^{UW}(a,b) = f(a).$$

On the other hand, since k_1 is decreasing and k_2 increasing, we have :

$$f'(a) = \frac{1}{(k_1(a) + k_2(a))^2} \left[k_1(a) k_2'(a) \theta_2 - k_2(a) k_1'(a) \eta_1 \right] \geq 0,$$

$$f'(b) = \frac{1}{(k_1(b) + k_2(b))^2} \left[k_1(b) k_2'(b) \eta_2 - k_2(b) k_1'(b) \theta_1 \right] \geq 0.$$

Moreover, since the function $q + K(g_1 - g_2)k_2$ is monotone and negative at the end points a,b, it is negative in the interval (a,b) and consequently the derivative f' is positive in the interval (a,b) and the function f is increasing in this interval. Therefore we have also :

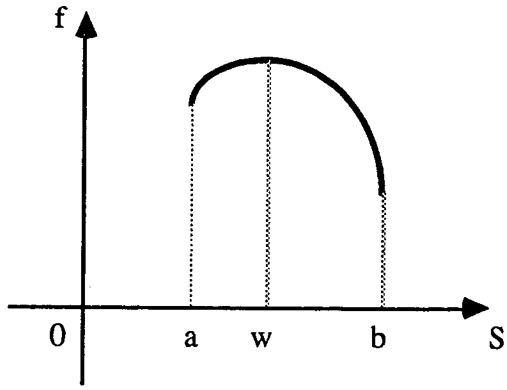
$$F^{G}(a,b) = F^{EO}(a,b) = f(a),$$

and all the flux functions give the same result.

The second case, $\theta_1 \leq 0 \leq \eta_2 \leq \theta_2$ is more difficult to handle since it involves many subcases. However we can show that there is no general ordering of the upstream weighted and Engquist-Osher flux functions. Whether one is larger than the other depends on the situation. Indeed, assume $q \equiv 0$ so

$$f = K(g_2 - g_1) \frac{k_1 k_2}{k_1 + k_2},$$

and assume we are in the situation described on the sketch below



where f is concave on (a,b) and maximum at a point w such that a<w<b. On one hand, from (4.12) we obtain

$$(4.13)\qquad F^{UW}(a,b) = K(g_2\text{-}g_1)\frac{k_1(b)k_2(a)}{k_1(b)+k_2(a)} = K(g_2\text{-}g_1)\frac{1}{\dfrac{1}{k_1(b)}+\dfrac{1}{k_2(a)}}.$$

We note that

$$F^{UW}(a,) < \operatorname*{Min}_{s\in(a,b)} f(s) = \operatorname{Min}\{f(a),f(b)\}.$$

On the other hand, definition (4.9) gives

$$F^{EO}(a,b) = f(a) - f(w) + f(b).$$

The end points (a,f(a)), (b,(f(b)) being fixed, if f(w) is large enough then $F^{EO}(a,b) \leq F^{UW}(a,b)$ and the numerical scheme associated to the two-phase upstream weighted numerical flux is less viscous. On the contrary, if f(w) is small enough (close enough to Max{f(a),f(b)}) then $F^{EO}(a,b) \geq F^{UW}(a,b)$ and the scheme associated to the Engquist-Osher numerical flux is less viscous.

Remark 4.1

This situation (q≡0) examplifies one of the specificities of the two-phase upstream weighted numerical flux. Even when f is monotone in [a,b], the calculation involves the points a and b (see expression 4.13). On the contrary standard upstream weighted numerical fluxes are equal to f(a) or f(b). The first numerical scheme is upstream with respect to the flow of each phases while the latter are upstream with respect to the derivative of f.

Remark 4.2

Still when q≡0, the multiphase upstream weighted scheme is based on the decomposition $1/f=1/k_1+1/k_2$ where k_1 is increasing and k_2 is decreasing while the Engquist-Osher scheme uses the decomposition $f=f_i+f_d$ where f_i is increasing and f_d is decreasing.

V. Higher order methods for multiphase flow

The study in the above sections gives some mathematical justification to the multiphase upstream weighted numerical flux widely used in reservoir simulation. Therefore, in order to minimize changes in the existing codes, it is reasonable to keep this numerical flux when designing more accurate methods. Following ideas due to Van Leer [13] a simple way to do so is to introduce a discontinuous piecewise linear approximation of the solution and a slope limitation device.

Precisely, the saturation of the phase ℓ in the interval $]x_{i-1/2},x_{i+1/2}[$ is now defined by its average value $S_{\ell,i}$ and its slope $\sigma_{\ell,i}$, $\ell=1,...,n$. Starting from a finite difference piecewise constant approximation $S^n_{\ell,i}$, the method has two steps. The first step constructs the slopes $\sigma^n_{\ell,i}$ in a way that prevents oscillations and the second step calculates the updated piecewise constant values $S^{n+1}_{\ell,i}$.

Step 1 : construction of the slopes

$$\sigma^n_{\ell,i}=\begin{cases} 0 \text{ if } S^n_{\ell,i}\le \mathrm{Min}(S^n_{\ell,i+1},S^n_{\ell,i-1}) \text{ or if } S^n_{\ell,i}\ge \mathrm{Max}(S^n_{\ell,i+1},S^n_{\ell,i-1}), \\ \mathrm{Min}[(S^n_{\ell,i}-S^n_{\ell,i-1})/h,\ (S^n_{\ell,i+1}-S^n_{\ell,i})/h] \text{ if } S^n_{\ell,i-1}\le S^n_{\ell,i}\le S^n_{\ell,i+1}, \\ \mathrm{Max}[(S^n_{\ell,i}-S^n_{\ell,i-1})/h,\ (S^n_{\ell,i+1}-S^n_{\ell,i})/h] \text{ if } S^n_{\ell,i-1}\ge S^n_{\ell,i}\ge S^n_{\ell,i+1}. \end{cases}$$

Step 2 : updating the new average values

$$\phi_i\frac{S^{n+1}_{\ell,i}-S^n_{\ell,i}}{\Delta t}+\frac{1}{h}\left(\varphi^n_{\ell,i+1/2}-\varphi^n_{\ell,i-1/2}\right)=0$$

where the approximate flow rate of the phase ℓ $\varphi^n_{\ell,i+1/2}$ is given in (1.6) with the mobilities defined in (1.7) or explicitely as in section II. However the saturation used to calculate the mobilities at $x_{i+1/2}$ are now the two limit values of the piecewise linear saturation at this point instead of the midpoint values.

Such a scheme can be justified along the lines of [9] in the case of two-phase flow (scalar case). It can be extended to multidimensional calculations through dimensional splitting or as a genuinely multidimensional scheme [1], [2].

VI. Conclusion

Upstream weighting for multiphase flow in reservoir simulation, though usually defined implicitely through simple physical considerations, can be expressed explicitely with a simple algorithm. The associate numerical fluxes have been shown to be Lipschitz-continuous (but no more), monotone and consistent. In the case of two-phase flow convergence follows and the two-phase upstream weighted numerical flux has been compared to Godunov's and Engquist-Osher's. It generates of course more viscosity than Godunov's but with respect to Engquist-Osher's it depends on the situation where it is so or not. Finally we showed how to design more accurate schemes while preserving the multiphase upstream weighted numerical flux.

REFERENCES

[1] G. Chavent, J. Jaffré,*"Mathematical Models and Finite Elements for Reservoir Simulation"*, (North Holland, Amsterdam, 1986).

[2] G. Chavent, J. Jaffré, R. Eymard, D. Guérillot, L. Weill, *"Discontinuous and mixed finite elements for two-phase in compressible flow"*, Paper SPE 16018 present at the 9th SPE Symposium on Reservoir Simulation, San Antonio, Texas (Feb.1-4, 1987).

[3] M.G. Crandall, A. Majda, *"Monotone difference approximations for scalar conservation laws"*, Math. Comp. 34 (1980), pp.1-21.

[4] B. Engquist, S. Osher,*"One-sided difference approximations for non linear conservation laws"*, Math. Comp. 36 (1981), pp321-351.

[5] S.K. Godunov, *"A finite difference method for the numerical computation of discontinuous solution of the equation of flow dynamics"*, Mat. Sb. 47 (1959), pp.271-290.

[6] A. Harten, P.D. Lax, B. Van Leer,*"On upstream differencing and Godunov-type schemes for hyperbolic conservation laws"*, SIAM Review 25, (1983), pp.35-61.

[7] N.N. Kuznetsov, S.A. Volosine,*"On monotone difference approximations for a first order quasi-linear equation"*, Soviet Math. Dokl. 17, (1976), pp.1203-1206.

[8] D.W. Peaceman,*"Fundamentals of Numerical Reservoir Simulation"*, Elsevier, New York, 1977.

[9] S. Osher,*"Convergence of generalized MUSCL schemes"*, SINUM 22 (1985), pp.947-961.

[10] P.H. Sammon,*"An analysis of upstream differencing"*, Soc. Pet. Eng. J. 3 (1988), pp.1053-1056.

[11] R. Sanders, *"On convergence of monotone finite difference schemes with variable spatial differencing"*, Math. Comp. 40 (1983), pp.91-106.

[12] E. Tadmor,*"Numerical viscosity and the entropy condition for conservative difference schemes"*, Math. Comp. 43 (1984), pp.369-381.

[13] B. Van Leer,*"Towards the ultimate conservative scheme : IV. A new approach to numerical convection"*, J. Comp. Phys. 23, (1977), pp.276-299.

Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique